# BLOCKING AND ANTI-BLOCKING PAIRS OF POLYHEDRA

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November 1970
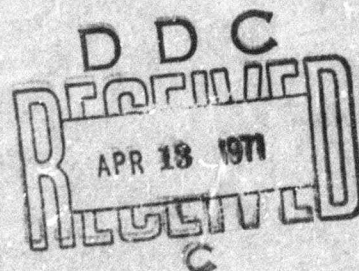
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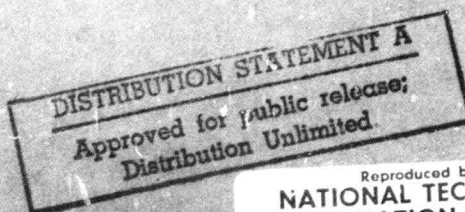
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# BLOCKING AND ANTI-BLOCKING PAIRS OF POLYHEDRA

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## SUMMARY

Some of the main notions and theorems about blocking pairs of polyhedra and anti-blocking pairs of polyhedra are described. The two geometric duality theories conform in many respects, but there are certain important differences. Applications to various combinatorial extremum problems are discussed, and some classes of blocking and anti-blocking pairs that have been explicitly determined are mentioned.

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## CONTENTS

# BLOCKING AND ANTI-BLOCKING PAIRS OF POLYHEDRA

## INTRODUCTION

There are several instances of combinatorial theorems dealing with maximum packing or minimum covering problems where theorems occur in dual pairs. Two specific examples, the first a packing example, the second a covering example, are the following.

Consider a two-terminal network and the families of paths joining the terminals and cuts separating the terminals. The maximum number of pairwise edge-disjoint paths joining the terminals is equal to the minimum number of edges in a cut separating the terminals [15]. Interchanging the roles of paths and cuts produces a dual theorem: the maximum number of pairwise edge-disjoint cuts separating the terminals is equal to the minimum number of edges in a path joining the terminals [24]. Matroid duality offers one explanation of the fact that these are equivalent theorems. Another explanation is afforded by the geometric theory of blocking pairs of polyhedra.

Consider a finite partially ordered set and the two families of chains and anti-chains of this partially ordered set. A well known theorem of Dilworth [4] states that the minimum number of chains required to cover all elements of the set is equal to the maximum number of elements in an anti-chain. Again interchanging the roles of chains and anti-chains produces a dual theorem: the minimum number

of anti—chains required to cover all elements of the set
is equal to the maximum number of elements in a chain.  The
latter theorem of the pair appears to be less well known;
it is certainly less meaty, since there is a very simple
construction which proves it.  And yet, from the viewpoint
afforded by the geometry of anti—blocking pairs of polyhedra,
these are equivalent theorems also.

   Another basic problem in extremal combinatorics that
the theory of blocking and anti—blocking pairs of polyhedra
partially elucidates is the following.  Let A be the m by n
incidence matrix of a family of subsets $S_1$, ..., $S_m$ of
$E = \{1, 2, ..., n\}$.  (For example, letting E be the edge
set of a graph G, the matrix A might specify the family of
tours of G, the family of paths joining two terminals in G,
the family of matchings of G, and so on.)  How does one
characterize the family as the extreme solutions of a system
of linear inequalities, i.e., how does one determine the
facets of a convex polyhedron having the row vectors of A
as its extreme points?  The most natural polyhedrom to look
at in this connection is the convex hull of the rows of A;
doing so might lead one to a consideration of anti—blocking
pairs of polyhedra.  Another polyhedron that might be
considered in this context is the vector sum of the convex
hull of the rows of A and the nonnegative orthant $R_+^n$.  Here
one might be led to a consideration of blocking pairs of
polyhedra.

The theory of anti–blocking pairs of polyhedra also bears on an unsolved problem in graph theory, the perfect graph conjecture due to Berge [1, 2, 3]. Indeed, a consequence of the theory is a theorem that appears to be a close relative of the perfect graph conjecture. We call it the pluperfect graph theorem.

In the body of the paper we describe some of the main notions and theorems concerning blocking pairs of polyhedra and anti–blocking pairs of polyhedra. The two duality theories conform in many respects, but there are certain important differences. No proofs are given; these will appear elsewhere [18, 19]. We also briefly mention some classes of blocking and anti–blocking pairs that have been explicitly determined and that are significant in extremal combinatorial theory.

## 1. THE BLOCKING RELATION

Let A be an m by n nonnegative matrix, and let w be a nonnegative n-vector. Consider the linear program

(1.1)
$$yA \leq w,$$
$$y \geq 0,$$
$$\max 1 \cdot y,$$

where $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$. We call (1.1) a "maximum packing program" or simply a "packing program." (Normally the word "packing" refers to a combinatorial situation in which A is a (0,1)-matrix, thought of as the incidence matrix of a family of subsets of $\{1, 2, \ldots, n\}$, w is an integer vector (usually $w = 1$), and the maximization in (1.1) is over all integer vectors y satisfying the constraints. It is, of course, a great simplification in this situation to drop the integer requirement on y, and to consider just the real (or rational) packing program (1.1).)

The linear programming dual of (1.1) is

(1.2)
$$Ax \geq 1,$$
$$x \geq 0$$
$$\max w \cdot x$$

The constraints in (1.2) describe an unbounded, n-dimensional, convex polyhedron

(1.3) $$\mathcal{B} = \{b \in R^n_+ | Ab \geq 1\}$$

situated in the nonnegative orthant $R^n_+$ of $R^n$. Polyhedron $\mathcal{B}$ is the vector sum of the convex hull of its extreme points and the nonnegative orthant:

(1.4) $$\mathcal{B} = \text{conv. hull } (\{b^1, \ldots, b^r\}) + R^n_+ ,$$

where $b^1, \ldots, b^r$ are the extreme points of $\mathcal{B}$.

Of course all rows of A may not represent facets of $\mathcal{B}$. It follows from the Farkas lemma that a row vector $a^i$ of A is inessential in defining $\mathcal{B}$ (i.e., does not represent a facet of $\mathcal{B}$) if and only if $a^i$ dominates (is greater than or equal to) a convex combination of other rows of A. If all rows of A are essential, we say that A is proper, and include among proper matrices the degenerate cases (i) A is a one-rowed zero matrix ($\mathcal{B}$ is empty), and (ii) A has no rows ($\mathcal{B} = R^n_+$).

Let

(1.5) $$\hat{\mathcal{B}} = \{a \in R^n_+ | a \cdot \mathcal{B} \geq 1\}.$$

We call $\hat{\mathcal{B}}$ the blocker of $\mathcal{B}$. Note that if $\mathcal{B}$ is empty, then $\hat{\mathcal{B}} = R^n_+$, and if $\mathcal{B} = R^n_+$, then $\hat{\mathcal{B}}$ is empty. Theorem 1 below shows that the blocking relation pairs members of the class of all convex polyhedra of type (1.3), and that nontrivial facets of one member of the dual pair of polyhedra represent extreme points of the other, and vice versa.

THEOREM 1. <u>Let the</u> m <u>by</u> n <u>matrix</u> A <u>be</u>
<u>proper with rows</u> $a^1$, ..., $a^m$. <u>Let</u> $\mathcal{B}$ =
$\{b \in R_+^n | Ab \geq 1\}$ <u>have extreme points</u> $b^1$, ..., $b^r$,
<u>let</u> B <u>be the matrix having rows</u> $b^1$, ..., $b^r$,
<u>and let</u> $\mathcal{A} = \{a \in R_+^n | Ba \geq 1\}$. <u>Then</u> (i) $\hat{\mathcal{B}} = \mathcal{A}$;
(ii) B <u>is proper and the extreme points of</u> $\mathcal{A}$
<u>are</u> $a^1$, ..., $a^m$; (iii) $\hat{\mathcal{A}} = \mathcal{B}$.

From (i) and (iii), $\hat{\hat{\mathcal{B}}} = \mathcal{B}$.

We call the matrix B of Theorem 1 the <u>blocking matrix</u>
or <u>blocker</u> of A. The blocking matrix of B is then A.

An example illustrating Theorem 1 in $R^2$ is shown in
Fig. 1.

It follows from Theorem 1 that if we are given the
matrix A, then there is a simple, easily described, but
very tedious algorithm for determining its blocking matrix
B.

Suppose that A is a (0,1)—matrix which is proper.
Then A is the m by n incidence matrix of a family consisting
of m pairwise noncomparable subsets of an n—set (the 1's
in a row of A do not contain those of any other row of A),
and conversely, such an A is proper. Such a family has
been called a clutter [14]. Thus Theorem 1 provides
information on the problem, mentioned in more general terms
in the Introduction, of characterizing the subsets comprising
an arbitrary clutter as the extreme points of a convex
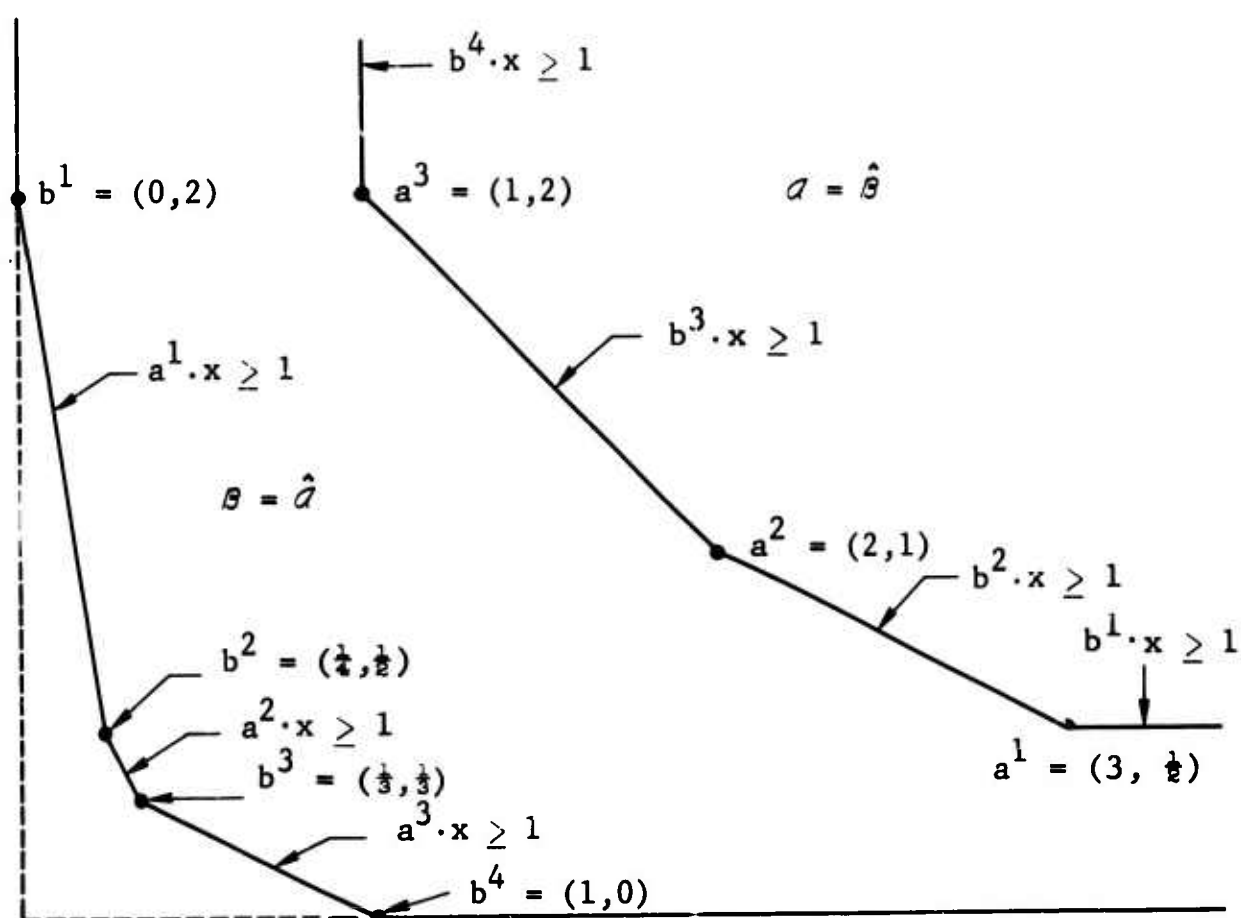
$$A = \begin{bmatrix} 3 & \frac{1}{2} \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 \\ \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \\ 1 & 0 \end{bmatrix}$$

Fig. 1.

polyhedron. If A is the incidence matrix of a clutter, then the incidence matrix b(A) of the blocking clutter has as its rows all (0,1)—vectors with n components that make inner product at least 1 with all rows of A and that are minimal with respect to this property [14, 23]. We have b(b(A)) = A. Each row of the matrix b(A) will be a row of the blocking matrix B of A, but B will usually have many other "fractional" rows. There are significant classes of clutters A for which B = b(A), however. For example, if A is the incidence matrix of all simple paths joining two terminals in a graph, then B = b(A), and B is consequently the incidence matrix of all simple cuts separating the terminals. (This is a consequence of Theorem 2.) Viewed in the context of Theorem 1 and this discussion, Lehman's interesting paper [22] is a study of clutters A for which B = b(A); his paper motivated much of our work.

## 2.  THE MAX-MIN EQUALITY AND MIN-MIN INEQUALITY

There are two characterizing properties of blocking pairs of matrices A, B that we shall describe.  In one of these, the matrices A and B play asymmetric roles; in the other they play symmetric roles.

Let A and B be m by n and r by n proper matrices having rows $a^1, \ldots, a^m$ and $b^1, \ldots, b^r$, respectively. Say that the max-min equality holds for the ordered pair A, B if and only if, for every $w \in R_+^n$, the packing program (1.1) has a solution y such that

$$(2.1) \qquad 1 \cdot y = \min_j b^j \cdot w.$$

Say that the min-min inequality holds for the unordered pair A, B if and only if, for every $\iota \in R_+^n$, $w \in R_+^n$, we have

$$(2.2) \qquad (\min_i a^i \cdot \iota)(\min_j b^j \cdot w) \leq \iota \cdot w.$$

Notice that the addition of inessential rows to either A or B affects neither the max-min equality nor the min-min inequality.

The max-min equality is the analogue of the max-flow min-cut equality for paths and cuts in a two-terminal network, and the min-min inequality is the analogue of the length-width inequality for two-terminal networks [5, 23].

THEOREM 2.  The max–min equality h    or
the ordered pair of proper matrices A,      d
only if A and B are a blocking pair.  Hence if
the max–min equality holds for A, B, it also
holds for B, A.

THEOREM 3.  The pair of proper matrices A,
B is a blocking pair if and only if (i) $a^i \cdot b^j \geq 1$
for all rows $a^i$ of A and $b^j$ of B, and (ii) the
min–min inequality holds for A, B.

These theorems can sometimes be used to prove that two
matrices A and B constitute a blocking pair.  For example, it
follows from Theorem 2 (or f  om Theorem 3) that the incidence
matrices A and B of minimal paths and minimal cuts in a two–
terminal network are a blocking pair.

If A is the incidence matrix of a clutter, it is natural
to ask for conditions under which the packing program (1.1)
always has integer solutions whenever w is an integer vector.
We say that the max–min equality holds strongly for A and its
blocking matrix B in this case.  (For example, if A and B are
the incidence matrices of minimal paths and minimal cuts
in a two–terminal network, the max–min equality holds strongly
for A, B, and for B, A.)  In view of (2.1), it is plausible,
and can be proved, that a necessary condition for the strong
max–min equality is that the blocking matrix B of A contain
no fractional rows, i.e., that B = b(A).  This condition is not
however, sufficient, as the example of Fig. 2 shows.

$$A: \begin{matrix} & 1\ 2\ 3\ 4\ 5\ 6 \\ & \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix} \end{matrix}$$

$$\text{(blocker)} \quad B: \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$
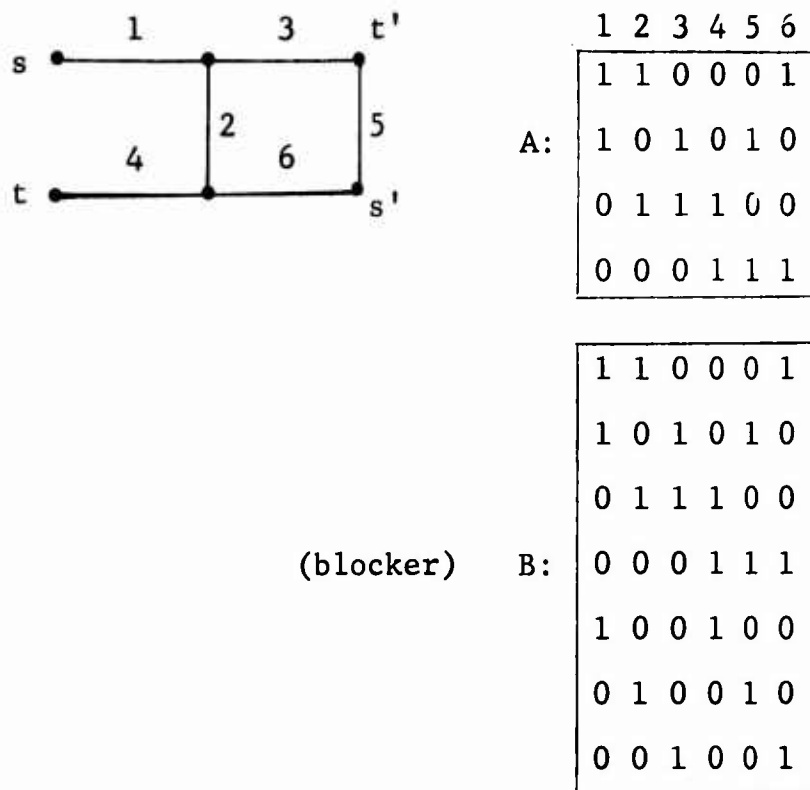
Fig. 2.

The matrix A of Fig. 2 is the incidence matrix of all s to
s' and t to t' paths in the graph shown there. It follows
from Theorem 2 and the "two-commodity" max-flow min-cut
theorem proved by Hu [21] that b(A), the incidence matrix
of all cuts separating s and t from s' and t', is the
blocking matrix B of A. But for w = 1, the unique solution
of the path-packing program (1.1) is $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. It
can also be shown for this example that the cut-packing
program $yB \leq w$, $y \geq 0$, max $1 \cdot y$, has an integer solution
whenever w is an integer vector. Thus, integer solutions

for one of the dual packing programs does not imply integer solutions for the other.  The question of whether the "two—commodity" cut—packing program always has integer solutions for integer weight vectors is open.

The phenomenon represented by the example of Fig. 2 does not arise for anti—blocking pairs of polyhedra (see Theorem 12).  This is one major difference between the two theories.

## 3. CONTRACTIONS AND DELETIONS

Let A be an m by n proper matrix. By a contraction
of coordinate i ∈ {1, 2, ..., n} in A we mean the following:
drop the i-th column of A and then drop all inessential
rows in the resulting matrix. A deletion of coordinate
i in A is the following: drop the i-th column of A, and
then drop all rows that had a positive entry in column i.
The new matrix obtained in each case is proper.

These operations are shown schematically in Fig. 3.
They are analogous to the operations of "contracting an
element" or " deleting an element" in a graph, matroid
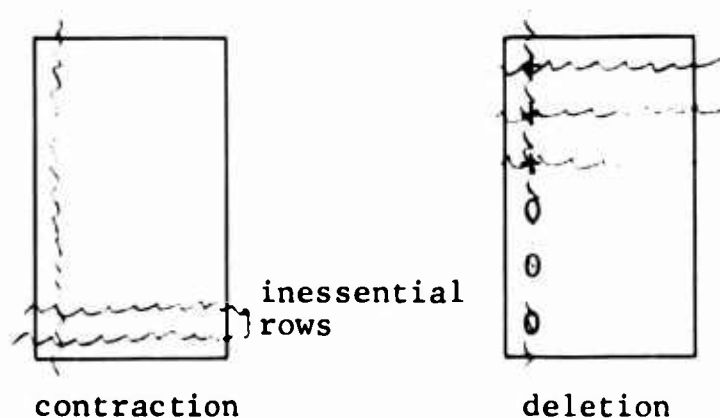[26], or clutter [23].

inessential rows

contraction          deletion

Fig. 3.

Geometrically, contracting coordinate i in A amounts to intersecting the polyhedron $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$ with the hyperplane $\xi_i = 0$; deleting coordinate i in A projects the polyhedron $\mathcal{B}$ orthogonally on the hyperplane $\xi_i = 0$. The two operations commute: contracting i, then deleting j, is equivalent to deleting j, then contracting i. Thus, one can define "minors" of A (or $\mathcal{B}$), just as in matroid theory, that arise by contracting some subset $I_1$ of coordinates and deleting some other subset $I_2$, since the order in which operations are carried out is immaterial. Let $A(I_1, I_2)$ denote the matrix obtained from A in this way.

> THEOREM 4. <u>Let A <u>and</u> B <u>be a blocking pair
> of matrices, and let</u> $I_1$ <u>and</u> $I_2$ <u>be disjoint subsets
> of the column index set of</u> A <u>and</u> B. <u>Then</u> $A(I_1, I_2)$
> <u>and</u> $B(I_2, I_1)$ <u>are a blocking pair.</u></u>

In other words, contractions and delections are dual operations. Theorem 4 can be used to prove the following "painting" theorem for blocking pairs.

> THEOREM 5. <u>Let A <u>and</u> B <u>be a blocking pair
> of matrices. For any partition of the column
> index set into two sets, blue and red, say,
> there is either a row of A whose support is blue
> or a row of B whose support is red, but not both.</u></u>

The analogous painting theorem is valid for a blocking pair of clutters. It in fact characterizes the blocking relation for clutters [14, 17].

## 4.  SOME CLASSES OF BLOCKING PAIRS

In addition to the primary example of paths and cuts
in a two-terminal network, there are other classes of
examples where blocking pairs of matrices have been
explicitly determined.  We discuss some of these, and
focus attention on the max-min equality, rather than the
min-min inequality, in doing so.

### 4.1.  Permutations

Let A be the incidence matrix of the clutter of all
k by k permutation matrices, viewed as vectors in $k^2$-space.
Thus A has size k! by $k^2$.  The blocking matrix B of A
consists of the essential rows of the following matrix $B^+$.
For each $I \subseteq \{1, 2, ..., k\}$, $J \subseteq \{1, 2, ..., k\}$ such that
$s(I, J) = |I| + |J| - k > 0$, let $b(I, J)$ be the $k^2$-vector
having components $1/s(I, J)$ for $i \in I$, $j \in J$, and zero
otherwise.  Let $B^+$ be the matrix consisting of all rows
$b(I, J)$.  That B is the blocking matrix of A follows from
Theorem 2 and results of [16].

Almost all of the rows of $B^+$ are essential.  In fact,
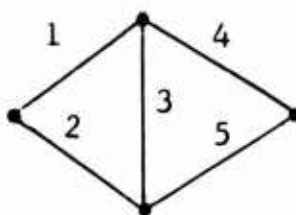the only inessential rows $b(I, J)$ are those where $I = \{1, ..., k\}$
and J is not a singleton, or $J = \{1, ..., k\}$ and I is not a
singleton.  This multiplicity of facets should be contrasted
with the few facets of the convex hull of all k by k (0,1)-
matrices having at most one 1 in each row and column, i.e.,
all permutations and subpermutations (see Sec. 7.1).

As shown in [16], there is an efficient algorithm for solving the packing program (1.1) for A. (We know of no efficient method for the dual packing program for B, although finding the minimizing row of A (i.e., finding $\min_i a^i \cdot w$) is the well known optimal assignment problem.) If the weight vector w has integer components, the algorithm of [16] can be used to find the best integer packing vector for the rows of A; this vector has component sum equal to $[\min_j b^j \cdot w]$, where $[\alpha]$ denotes the largest integer less than or equal to $\alpha$.

## 4.2. Matroid Bases

Let A be the m by n incidence matrix of all bases in a matroid on $\{1, 2, \ldots, n\}$. Thus, for example, A might be the incidence matrix of all spanning trees in a connected graph having edge set $\{1, 2, \ldots, n\}$. (See Fig. 4.) It can be deduced from results of Tutte [25] and of Edmonds [8, 9] that the blocking matrix B of A consists of the essential rows of the matrix $B^+$ which has a row $b(\overline{S})$ corresponding to each nonempty complement $\overline{S}$ of a span (closed set) S: the row $b(\overline{S})$ has components $1/(\text{rank}(\{1, 2, \ldots, n\}) - \text{rank}(S))$ in positions corresponding to elements of $\overline{S}$, zeros elsewhere. (See Fig. 4.) The packing problem (1.1) for A, in either integer or rational form, has been solved: Edmonds has described an efficient algorithm for finding the maximum number of disjoint matroid bases, and this can be extended to the case of a rational weight function w.

The packing problem for B has also been solved. Finding a minimizing row of A can be accomplished by the greedy algorithm [10], and it is not hard to describe a method of determining a maximum packing vector y for $B^+$. Here contractions and deletions can be brought into play in determining y.



$$
A: \quad
\begin{array}{c c c c c}
1 & 2 & 3 & 4 & 5 \\
\hline
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
\hline
\end{array}
\qquad
B: \quad
\begin{array}{c c c c c}
1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\hline
\end{array}
$$

Fig. 4.

## 4.3. Lines in an Array

Call a row or column of an r by s array a "line", and let A be the r + s by rs incidence matrix of all lines in such an array. Matrix A is totally unimodular, and hence its blocking matrix B is a (0,1)-matrix: B = b(A). Each row of B corresponds to an r by s (0,1)-matrix that has at least one 1 in each line, which is minimal with respect to this property.

The max-min equality holds strongly for both A, B and for B, A. The first assertion can be viewed as a consequence of the fact that A is totally unimodular. Efficient algorithms can also be described which produce integer answers in both packing programs. The strong max-min equality for B, A implies the following theorem about bipartite graphs: the maximum number of edge-disjoint covers (of vertices by edges) in a bipartite graph is equal to the minimum valence in the graph. This is a companion to the König theorem that the minimum number of colors required to color the edges of a bipartite graph is equal to the maximum valence in the graph (see Sec. 7.1).

## 4.4. Spanning Arborescences

Let G be a directed graph having edge set {1, 2, ..., n} and let A be the m by n incidence matrix of all spanning arborescences of G that are rooted at a particular vertex v of G. (A spanning arborescence rooted at v is a spanning tree of the underlying undirected graph having the properties

(i) each vertex of G other than v has just one edge of the
arborescence directed toward it; (ii) no edge of the
arborescence is directed toward the root v.)  Thus we could
suppose that all edges of G are directed away from v.

Edmonds [11] has described an efficient algorithm for
computing a minimum or maximum weight arborescence in G,
and has determined inequalities that define the convex hull
of the incidence vectors of all branchings in G.  (A branching
is a forest whose edges are directed toward different vertices.)
The algorithm of [11] can be modified to prove (i) the blocking
matrix B of A is a $(0,1)$-matrix, so that $B = b(A)$, and (ii)
the max—min equality holds strongly for B, A.  The matrix B
can be described as the incidence matrix of all minimal cuts
directed away from the root v (see Fig. 5).  That is, consider
any partition of the vertices of G into two sets S and $\overline{S}$, and
suppose $v \in S$.  By a <u>cut directed away from</u> v we mean the set
of edges of G directed from S to $\overline{S}$.

Since the max—min equality also holds for the $(0,1)$—
matrices A, B, it is natural to ask if it holds strongly
in this direction as well.  Edmonds has proved that it does
[12, 13].

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*In terms of the discussion in the Introduction, Edmonds'
results on optimum branchings would appear to fall more
naturally in the domain of anti—blocking pairs of polyhedra.
Indeed, the anti—blocking polyhedron of the convex hull of
all branchings is determined explicitly in [11], and the dual
variables there solve the relevant covering problem, not the
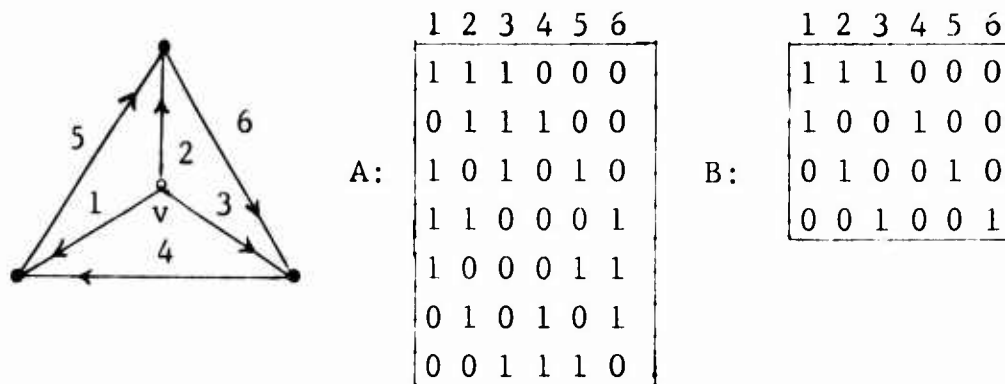packing problem for cuts directed away from the root.

$$
A: \begin{array}{cccccc}
 & 1\ 2\ 3\ 4\ 5\ 6 \\
\hline
\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}
\end{array}
\qquad
B: \begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
$$

Fig. 5.

It should be emphasized that if the incidence matrix A is expanded to include all spanning arborescences of G, rather than just those rooted at a particular vertex, then the blocking matrix B of A may not be a (0,1)–matrix. (See Fig. 6 for a simple example showing this.)



$$
A: \begin{array}{ccc}
1 & 2 & 3 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\qquad
B: \begin{array}{ccc}
1 & 2 & 3 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
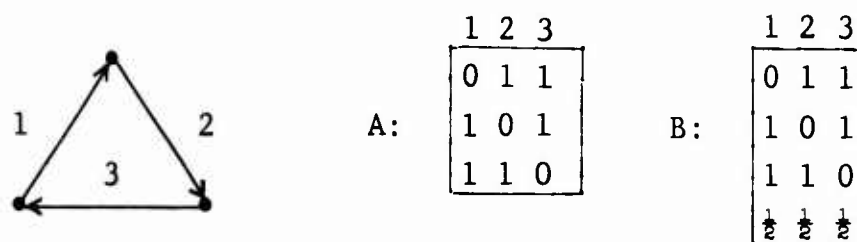\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
$$

Fig. 6.

## 4.5. Complementary Orthogonal Subspaces

Let $V$ and $V^\perp$ be complementary orthogonal subspaces of $R^n$, and let $E$ and $E^\perp$ be the sets of all elementary vectors of $V$ and $V^\perp$, respectively. (A vector of $V$ is elementary if it is nonzero and has minimal support over all nonzero vectors of V.) Define

$$E_1 = \{a = (\alpha_1, \ldots, \alpha_n) \in V | \alpha_1 = 1\},$$

$$E_1^\perp = \{b = (\beta_1, \ldots, \beta_n) \in V^\perp | \beta_1 = 1\}.$$

The sets $E_1$ and $E_1^\perp$ are finite, say

$$E_1 = \{a^1, \ldots, a^m\},$$

$$E_1^\perp = \{b^1, \ldots, b^r\}.$$

For each $a^i = (1, \alpha_2^i, \ldots, \alpha_n^i) \in E_1$, $b^j = (1, \beta_2^j, \ldots, \beta_n^j) \in E_1^\perp$, let

$$\bar{a}^i = (|\alpha_2^i|, \ldots, |\alpha_n^i|),$$

$$\bar{b}^j = (|\beta_2^j|, \ldots, |\beta_n^j|),$$

and let A and B be the nonnegative matrices with $n-1$ columns having rows $\bar{a}^i$ and $\bar{b}^j$, respectively. The matrices A and B constitute a blocking pair. This follows from the proof in [17] that the max–min equality holds for A, B.

If the space V (and hence $V^{\perp}$) is regular [26], that
is, if V can be viewed as the row space of a totally
unimodular matrix, then each elementary vector of $E_1$ (and
of $E_1^{\perp}$) has components 0, 1 or −1, and hence the matrices
A and B are (0,1)−matrices.  (For example, if V is the
row space of the (0,±1)−vertex−edge matrix of an oriented
graph, this construction yields A as the incidence matrix
of all (minimal) cuts separating the two ends of edge 1 in
the underlying unoriented graph with edge 1 suppressed,
and B as the incidence matrix of all (minimal) paths
joining these two vertices in the same graph.)  Results
of [17] then imply that the max-min equality holds strongly
for A, B and B, A.

## 5. THE ANTI-BLOCKING RELATION

Let A be an m by n nonnegative matrix. We assume that $m \cdot 1$ and that no column of A consists entirely of zeros. Consider the linear program

$$(5.1) \qquad \begin{aligned} yA &\geq w, \\ y &\geq 0, \\ \min 1 \cdot y, \end{aligned}$$

where w is a nonnegative n–vector. We call (5.1) a "minimum covering program" or simply a "covering program" for A and the weight vector w. The linear programming dual of (5.1) is

$$(5.2) \qquad \begin{aligned} Ax &\leq 1, \\ x &\geq 0, \\ \max w \cdot x. \end{aligned}$$

Let

$$(5.3) \qquad \mathcal{B} = \{b \in R_+^n \,|\, Ab \leq 1\}$$

be the n–dimensional convex polyhedron corresponding to the dual constraints (5.2). The assumption that A has a positive entry in each column is equivalent to saying that $\mathcal{B}$ is bounded. Thus $\mathcal{B}$ is the convex hull of its extreme points $b^1, \ldots, b^r$:

(5.4) $\qquad \beta = \text{conv. hull } (\{b^1, \ldots, b^r\}).$

(In terms of the covering program (5.1), it is not restrictive to assume that $\beta$ is bounded, since if A has a column of zeros, (5.1) is infeasible unless the corresponding component of w is zero, in which case this column can be ignored.)

The Farkas lemma implies that a row vector $a^i$ of A is inessential in defining $\beta$ if and only if $a^i$ is dominated by a convex combination of other rows of A.

Let

(5.5) $\qquad \bar{\beta} = \{a \in R_+^n | a \cdot \beta \leq 1\}.$

We call $\bar{\beta}$ the anti—blocker of $\beta$.

THEOREM 6. Let A be a nonnegative matrix having no zero columns and suppose $\beta = \{b \in R_+^n | Ab \leq 1\}$ has extreme points $b^1, \ldots, b^r$. Let matrix B have rows $b^1, \ldots, b^r$. Then B is nonnegative, has no zero columns, and (i) $\bar{\beta} = \{a \in R_+^n | Ba \leq 1\}$, (ii) $\bar{\bar{\beta}} = \beta$.

An example illustrating Theorem 6 is shown in Fig. 7.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
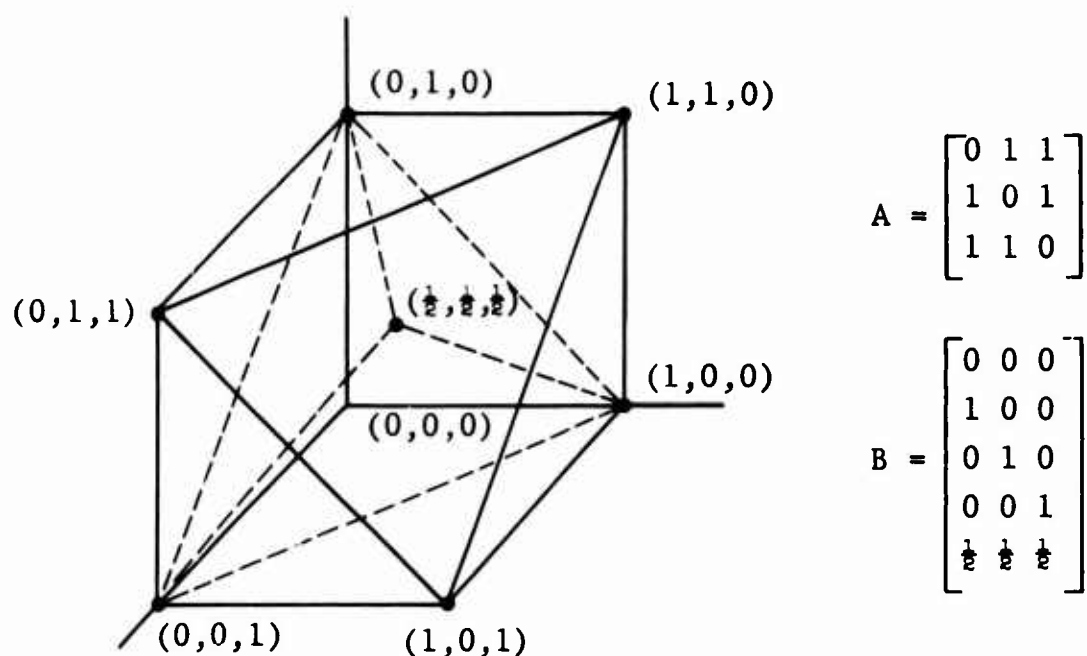
Fig. 7.

In the example, if we start with the matrix A, all of whose rows are essential for $\mathcal{B}$ (define facets of $\mathcal{B}$), and compute the extreme points of $\mathcal{B}$, we obtain the matrix B. All rows of B except the first are essential for $\overline{\mathcal{B}}$. On the other hand, if we start with B (or the essential rows of B) and compute the extreme points of $\overline{\mathcal{B}}$, we obtain the rows of A and the inessential rows (0,0,0), (1,0,0), (0,1,0), (0,0,1). Notice that an inessential extreme point (say an extreme point of $\overline{\mathcal{B}}$ that does not represent a facet of $\mathcal{B}$) is a projection of some other essential extreme point. This is true in general.

THEOREM 7.  Let A be a nonnegative matrix defining the polyhedron $\mathcal{B} = \{b \in R_+^n | Ab \leq 1\}$ and let b, $b^1$, ..., $b^s$ be points of $\mathcal{B}$ such that b is an extreme point of $\mathcal{B}$ and is dominated by a convex combination of $b^1$, ..., $b^s$.  Then b can be obtained from some $b^i$ by setting certain components of $b^i$ equal to zero.


Since there is not a 1–1 correspondence between facets and extreme points for anti–blocking pairs of polyhedra, we shall not insist on dealing with "proper" matrices A and B defining them.  Rather we shall say that any pair of matrices A and B that define an anti–blocking pair of polyhedra constitute an anti–blocking pair of matrices.

Theorems 6 and 7 imply:


THEOREM 8.  Let A be the m by n incidence matrix of a clutter of m subsets $S_1$, ..., $S_m$ of $\{1, ..., n\}$, and suppose A has no zero columns.  Let B be an anti–blocking matrix of A.  Then the extreme points of the bounded polyhedron $\mathcal{Q} = \{a \in R_+^n | Ba \leq 1\}$ are precisely the rows of A together with all incidence vectors of subsets of $S_1$, ..., $S_m$ (i.e., all projections of rows of A).

The example of Fig. 7 illustrates Theorem 8.

The matrix B of Theorem 8 contains a row corresponding to each (0,1)—vector that makes inner product at most 1 with every row of A and that is maximal with respect to this property. But if we denote the set of such rows by a(A), it is not true in general that a(a(A)) = A. Indeed, this relation holds if and only if A is the incidence matrix of the family of maximal cliques of a graph G, in which case a(A) is the family of maximal independent sets of vertices of G.

Notice in the example of Fig. 7 that if the polyhedra $\mathcal{B}$ and $\overline{\mathcal{B}}$ are projected orthogonally on the hyperplane $\xi_1 = 0$, say, the resulting polyhedra again constitute an anti—blocking pair. This is true in general. In terms of an anti—blocking pair A, B of matrices, if the i—th column of A is dropped, then an anti—blocking matrix of the resulting matrix is obtained from B by dropping its i—th column. The situation is simpler than the analogous one for blocking pairs (see Sec. 3).

We conclude this section with a theorem that describes, in the context of (0,1)—matrices, a connection between blocking and anti—blocking pairs.

THEOREM 9. Let A be the m by n incidence matrix of a clutter on {1, ..., n} and suppose A has no column consisting of 1's. Let B be the r by n blocking matrix of A, and let $\rho_j$

denote the sum of the elements in the j–th row $b^j$ of B. Let A' be the complement of A. Then an anti–blocking matrix B' of A' is the r + n by n matrix having rows $b^1/(\rho_1 - 1)$, ..., $b^r/(\rho_r - 1)$, $e_1$, ..., $e_n$, where $e_i$ is the i–th unit vector.

The example of Fig. 8 illustrates Theorem 9. The matrix A of Fig. 8 is the incidence matrix of all spanning trees of the graph shown there, A' is the incidence matrix of all cotrees, and B' is obtained from B as outlined in Theorem 9. Inessential rows of B' are crossed out in the figure.
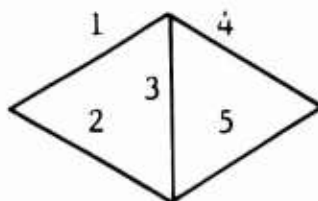
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad (\geq 1)$$

$$B' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (\leq 1)$$

Fig. 8.

## 6. THE MIN–MAX EQUALITY AND MAX–MAX INEQUALITY

Let A and B be m by n and r by n nonnegative matrices having rows $a^1$, ..., $a^m$ and $b^1$, ..., $b^r$, respectively, and suppose neither A nor B has zero columns. Say that the min–max equality holds for the ordered pair A, B, if and only if, for every $w \in R_+^n$, the covering program (5.1) has a solution y such that

$$(6.1) \qquad 1 \cdot y = \max_j b^j \cdot w.$$

Say that the max–max inequality holds for the unordered pair A, B, if and only if, for every $\ell \in R_+^n$, $w \in R_+^n$, we have

$$(6.2) \qquad (\max_i a^i \cdot \ell)(\max_j b^j \cdot w) \geq \ell \cdot w.$$

Again the addition of inessential rows to A or to B affects neither the min–max equality nor max–max inequality.

THEOREM 10. The min–max equality holds for the ordered pair A, B, if and only if A and B are an anti–blocking pair. Hence if the min–max equality holds for A, B, it also holds for B, A.

THEOREM 11. The pair A, B, is an anti–blocking pair if and only if (i) $a^i \cdot b^j \leq 1$

for all rows $a^i$ of A and $b^j$ of B, and (ii) the max max inequality holds for A, B.

If A is a (0,1) matrix with anti–blocker B, we say that the min–max equality holds strongly for A, B, provided the covering program (5.1) has an integer solution vector y wherever w is a nonnegative integer vector. In contrast with the situation for blocking pairs, one can prove

> THEOREM 12. Let A be a (0,1)–matrix and let B be an anti–blocking matrix of A. The min–max equality holds strongly for A, B if and only if each essential row of B is a (0,1)–vector. Hence if the min–max equality holds strongly for A, B, it also holds strongly for B*, A, where B* consists of the essential rows of B.

A consequence of Theorem 12 is that if the min–max equality holds strongly for the incidence matrix A of a clutter and an anti–blocking matrix B of A, then A is the incidence matrix of the family of maximal cliques of a graph G, and B* = a(A) is the incidence matrix of the family of maximal independent sets of vertices of G. Theorem 12 is intimately related to the perfect graph conjecture due to Berge; this will be discussed in more detail in Sec. 8.

## 7. SOME CLASSES OF ANTI–BLOCKING PAIRS

Anti–blocking matrices have been explicitly determined
for several classes of (0,1)–matrices. In describing some
of these, we again focus primary attention on the min–max
equality, rather than the max–max inequality.

### 7.1. Permutations

Let A be the $m = k!$ by $n = k^2$ (0,1)–matrix having a
column for each cell $ij$ of a $k$ by $k$ array and having a row
corresponding to each $k$ by $k$ permutation matrix, viewed as
a vector in $R^n$. It is well known that the inequalities

$$(7.1) \qquad \sum_j \xi_{ij} \leq 1, \quad \sum_i \xi_{ij} \leq 1, \quad \xi_{ij} \geq 0,$$

have as extreme solutions $x = (\xi_{ij})$ precisely the rows of
A and all projections of these rows. In other words, an
anti–blocking matrix of A is the $r = 2k$ by $n = k^2$ matrix
B whose rows are the incidence vectors of all lines of the
$k$ by $k$ array. This is a consequence of the fact that B is
totally unimodular, from which it also follows that the
min–max equality holds strongly for B, A and hence (by
Theorem 12) strongly for A, B. For w a (0,1)–vector, the
strong min–max equality for B, A is the classical König
theorem on maximum matchings and minimum covers in bipartite
graphs, and the strong min–max equality for A, B is the
theorem, also due to König, that the minimum number of

colors required for an edge—coloring in a bipartite graph is equal to the maximum valence in the graph.

Matrix B is the clique matrix of the line graph of the complete bipartite graph $G_{k,k}$.

## 7.2.  Chains in a Partially Ordered Set

Let A be the incidence matrix of all maximal chains in a partially ordered set.  An anti—blocking matrix B of A is the incidence matrix of all maximal anti—chains of the partially ordered set.  Neither A nor B is totally unimodular in general.

For w a (0,1)—vector, the strong min—max equality for A, B is the Dilworth theorem [4]; the extension to nonnegative integer vectors w can be deduced from the Dilworth theorem by replicating elements appropriately.  The strong min—max equality for B, A is, on the other hand, a triviality:  for w a (0,1)—vector, it asserts that the minimum number of anti—chains to cover all elements of a partially ordered set is equal to the length of a longest chain, a fact that is easily proved; again the extension to nonnegative integer vectors w can be deduced by replicating elements appropriately.

Matrix A is the clique matrix of a comparability graph.

## 7.3.  Matchings

Let A be the incidence matrix of the family of all matchings in a graph.  Edmonds has proved [6, 7] that inequalities of two types characterize the convex hull of
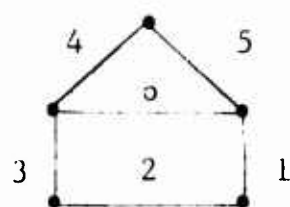
the rows of A.  Let $\xi_{ij}$ be a nonnegative variable associated with the edge ij having end—vertices i and j in the graph G having s vertices.  The inequalities can be written as

$$(7.2) \qquad \sum_{j \in N_i} \xi_{ij} \leq 1, \quad i = 1, \ldots, s,$$

$$(7.3) \qquad \sum_{i \in 0, j \in 0} \xi_{ij} \leq \frac{|0| - 1}{2}, \quad \text{all } 0 \subseteq \{1, \ldots, s\}.$$

In (7.2) $N_i$ denotes the set of vertices that neighbor i; in (7.3) the subset of vertices 0 has odd cardinality $|0|$, and the sum is over all edges joining members of 0. Thus (7.2) and (7.3) determine an anti—blocking matrix B of A.  (See Fig. 9 for an example; only essential rows are shown there.)  Edmonds' proof that (7.2) amd (7.3) have just the matchings as extreme solutions is based on his matching algorithm for solving the dual programs $Bx \leq 1$, $x \geq 0$, max $w \cdot x$, and $yB \geq w$, $y \geq 0$, min $1 \cdot y$, thereby establishing the min—max equality for B, A.

Note that the best integer answer in the covering program for A and $w = 1$ provides a coloring of the edges of the graph with the least member of colors.  Hence the integer form of this problem is unsolved.

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A:

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
$$

B:

$$
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
$$

Fig. 9.

## 7.4. Independent Sets in Matroids

Let A be the incidence matrix of the family of independent sets in a matroid on $\{1, \ldots, n\}$, and let $\xi_i$ be a nonnegative variable associated with $i \in \{1, \ldots, n\}$. It has been proved by Edmonds [10] that the inequalities

$$(7.4) \qquad \sum_{i \in S} \xi_i \leq \text{rank }(S), \qquad \text{all spans } S \subseteq \{1, \ldots, n\},$$

have precisely the rows of A as extreme solutions. Thus
(7.4) determines an anti-blocking matrix B of A. (This
can also be deduced from results described in Sec. 4.2,
Theorem 9, and the fact that the complement of a base in
a matroid is a base in the dual matroid.) Edmonds has
also shown that the best integer answer in the covering
program for A and w has component sum equal to $\langle \max_S w \cdot b^S \rangle$,
where $\langle \alpha \rangle$ denotes the least integer greater than or equal
to $\alpha$, and $b^S$ denotes the row of B corresponding to span
S [8].

## 7.5. Sets that are Independent in Two Matroids

Consider two matroids defined on the same set of
elements $\{1, \ldots, n\}$, and let A be the incidence matrix
of the family of sets that are independent in both matroids.
Let $r_1(X)$ $(r_2(X))$ denote the rank of set $X \subseteq \{1, \ldots, n\}$
in the first (second) matroid. Edmonds has proved that
the inequalities

$$(7.5) \qquad \sum_{i \in X} \xi_i \leq \min(r_1(X), r_2(X)), \qquad \text{all } X \subseteq \{1, \ldots, n\},$$

in nonnegative variables $\xi_i$ have just the rows of A as
extreme solutions [12]. Thus (7.5) determines an anti-
blocking matrix B of A.

If $\{1, \ldots, n\}$ is the set of edges of a directed
graph G, then the incidence matrix A of the family of

branchings (see Sec. 4.4) in G is the incidence matrix of
the family of sets that are independent in two matroids
which can be associated with G:  one of these is the
graphic matroid of the underlying undirected graph, and
the other is the partition matroid defined on the edges
of G by saying that a set of edges is independent if and
only if each edge of the set is directed toward a different
vertex of G.  The inequalities (7.5) can be simplified in
this case [11].

Efficient algorithms have been devised by Edmonds
[12] for solving the covering program $yB \geq w$, $y \geq 0$, min $1 \cdot y$
and its dual linear program $Bx \leq 1$, $x \geq 0$, max $w \cdot x$,
thereby establishing the min–max equality for B, A.  Another
efficient algorithm for this class of problems has been
described by Lawler [22].

## 8. THE PERFECT GRAPH CONJECTURE

Let G be a graph and let $\gamma(G)$, $\lambda(G)$, $\pi(G)$, and $\omega(G)$ denote respectively the chromatic number of G (the minimum number of independent sets of vertices that cover all vertices of G), the clique number of G (the size of a largest clique in G), the partition number of G (the minimum number of cliques that cover all vertices of G), and the independence number (stability number) of G (the size of a largest independent set in G). The graph G is $\gamma$-perfect if $\gamma(H) = \lambda(H)$ for every vertex-generated sub-graph H of G; G is $\pi$-perfect if $\pi(H) = \omega(H)$ for every vertex-generated subgraph H of G; G is perfect if it is both $\gamma$-perfect and $\pi$-perfect, i.e., if both G and its complementary graph $\overline{G}$ are $\gamma$-perfect. The perfect graph conjecture due to Berge [1, 2, 3] asserts that $\gamma$-perfection implies perfection. Among the several classes of graphs that are known to be perfect are rigid-circuit graphs and comparability graphs. The latter class includes bipartite graphs.

Now let A and B be the m by n and r by n incidence matrices of the families of (maximal) cliques and (maximal) independent sets, respectively, of G, where A has rows $a^1, \ldots, a^m$ and B has rows $b^1, \ldots, b^r$. Define functions $\gamma_G(w)$, $\lambda_G(W)$, $\pi_G(w)$, $\omega_G(w)$, where w is a nonnegative integer n-vector, as follows. Let $\gamma_G(w)$ be the minimum in the integer program $yB \geq w$, $y \geq 0$, min $1 \cdot y$ (i.e., y is restricted

to integer vectors); let $\lambda_G(w) = \max_i a^i \cdot w$; let $\pi_G(w)$ be

the minimum in the integer program $yA \geq w$, $y \geq 0$, min $1 \cdot y$;

and let $\omega_G(w) = \max_j b^j \cdot w$. Let us say that G is $\gamma$-pluperfect

if $\gamma_G(w) = \lambda_G(w)$ for all nonnegative integer vectors w,

G is $\pi$-pluperfect if $\pi_G(w) = \omega_G(w)$ for all such w, and G

is pluperfect if it is both $\gamma$-pluperfect and $\pi$-pluperfect.

Note that $\gamma$-perfection, for example, requires only that

$\gamma_G(w) = \lambda_G(w)$ for all $(0,1)$-vectors w, rather than for all

nonnegative integer vectors w.

Theorem 12 implies


THEOREM 13.  If G is $\gamma$-pluperfect, then

G is pluperfect.


Actually Theorems 12 and 13 are equivalent.

Thus to prove the perfect graph conjecture, it would

suffice to prove that $\gamma$-perfection implies $\gamma$-pluperfection.

For this it would suffice to show that if G is $\gamma$-perfect,

and if we duplicate an arbitrary vertex v in G and join

⟩ its duplicate vertex, the new graph G' is again

perfect.

Among the classes of graphs that are known to be

pluperfect are rigid-circuit graphs and comparability

graphs (Sec. 7.2).

Our final theorem lists some equivalent versions of

the perfect graph conjecture.

THEOREM 14. The following statements are equivalent:

(i)  If G is $\gamma$-perfect, then G is perfect.

(ii)  If G is $\gamma$-perfect, then G is pluperfect.

(iii)  Let A be a (0,1)-matrix such that the covering program $yA \geq w$, $y \geq 0$, min $1 \cdot y$ has an integer solution vector y whenever w is a (0,1)-vector.  Then this program has an integer solution vector y whenever w is a nonnegative integer vector.

It is our feeling that Theorem 14 casts some doubt on the validity of the perfect graph conjecture.

## 9. REMARKS AND QUESTIONS

The fact that blocking or anti-blocking matrices have been expressly uncovered for several significant classes of (0,1)-matrices, together with, in many instances, interesting and efficient methods for solving the associated packing or covering problems, is an important one which deserves more emphasis than we have given it in Secs. 4, 7, and 8. These classes of examples provide the substance without which the present theory would be largely devoid of interest.

Figure 10 lists some classes of clutters and indicates the present state of knowledge about blocking or anti-blocking polyhedra for each: an entry "0" in the table of Fig. 10 means that not much is known (at least to the author) about the blocker or anti-blocker, a "1" that the blocker or anti-blocker has been determined.

Consider the first two lines of Fig. 10. The blocking matrix for the incidence matrix of the clutter of simple paths joining two terminals in a graph is the incidence matrix of all simple cuts separating the terminals. While an anti-blocking matrix is unknown in general, there is good reason to believe that it is considerably more complicated in structure than the blocking matrix. On the other hand, both the blocker and anti-blocker are known for the permutations, and the blocker is more complicated in structure. Thus it may be the case for some of the lines

| Clutter | Blocker | Anti-Blocker |
|---|---|---|
| two—terminal paths | 1 | 0 |
| permutations | 1 | 1 |
| lines in array | 1 | 1 |
| maximal chains in p.o. set | 1 | 1 |
| maximal matchings | 0 | 1 |
| matroid bases | 1 | 1 |
| spanning arboresences with fixed root | 1 | 0 |
| maximal sets independent in two matroids | 0 | 1 |
| tours | 0 | 0 |
| minimal sets of vertices that cover edges | 0 | 0 |
| maximal cliques | 0 | 0 |
| minimal strongly connected subgraphs | 0 | 0 |

Fig. 10.

where neither is known that one of the two indicated problems is considerably easier than the other. At any rate, turning a "0" into a "1" in Fig. 10, or learning more about the relevant packing (covering) problem, or the associated minimum (maximum) weight problem, would be of considerable interest. For example, a very natural problem suggested by the last line of Fig. 10 is that of constructing a minimum

weight strongly connected subgraph of a directed graph G,
each of whose edges has a nonnegative weight.  This is an
analogue for directed graphs of the minimum spanning tree
problem for undirected graphs.  In practical terms, the
problem is to find the cheapest network of one—way streets
joining pairs of points so that it is possible to get from
any point to any other.  For this problem, information on
the blocking matrix B of the incidence matrix A of the
family of minimal strongly connected subgraphs of G would
be desirable.  The incidence matrix b(A) of the blocking
clutter is the matrix of all minimal directed cuts of G,
but B ≠ b(A), since, for example, the blocking matrix of
b(A) for the gra₊ ⸴ shown in Fig. 11 contains the fractional
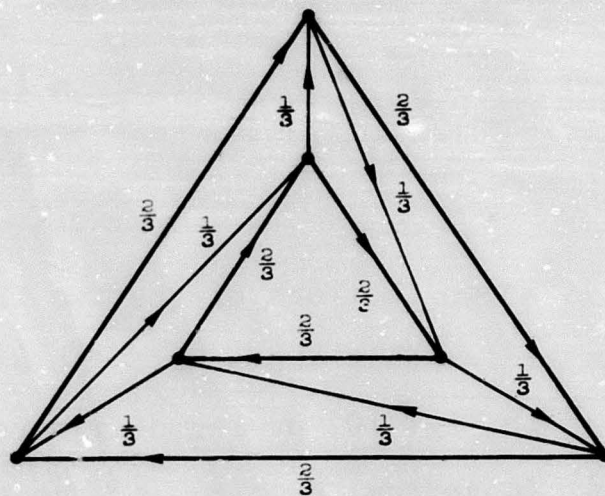row indicated schematically there.



Fig. 11.

In connection with some of the other lines in Fig. 10, we ask some specific questions.

## 9.1.  Tours

Let G be a complete undirected graph with edge set $E = \{1, \ldots, n\}$ and let A be the m by n incidence matrix of the clutter of tours (Hamiltonian circuits) $T_1, \ldots, T_m$ of G.  Thus if G has k vertices, then $n = \binom{k}{2}$, $m = \frac{1}{2}(n-1)!$. Do the inequalities in nonnegative variables $\xi_j$, $j \in E$,

$$(9.1) \qquad \sum_{j \in X} \xi_j \geq \min_{1 \leq i \leq m} |T_i \cap X|, \quad \text{all } X \subseteq E,$$

have just the rows of A as extreme solutions?  For $k \leq 5$, the answer appears to be in the affirmative, but it seems unlikely to be so for $k > 5$.

## 9.2.  Tait Bicolorings

In Sec. 7.3 we noted in connection with Edmonds' work on matchings that if A is the incidence matrix of the family of matchings in a graph, then the best integer answer in the covering program for A and $w = 1$ yields an edge coloring (Tait coloring) with the least number of colors.  If we permit "fractional colorings," i.e., if we consider the linear program $yA \geq 1$, $y \geq 0$, min $1 \cdot y$, over the rationals, then Edmonds' results and Theorem 10 show what the answer is.  For example, one can deduce that if G is a 3−valent,

3 connected graph, then min 1·y = 3.  For such a graph,
does this program always have a solution y having components
0, 1, or ½?  Put another way, can one assign pairs of colors
from six colors to the edges of such a graph so that all
six colors appear at each vertex, i.e., does such a graph
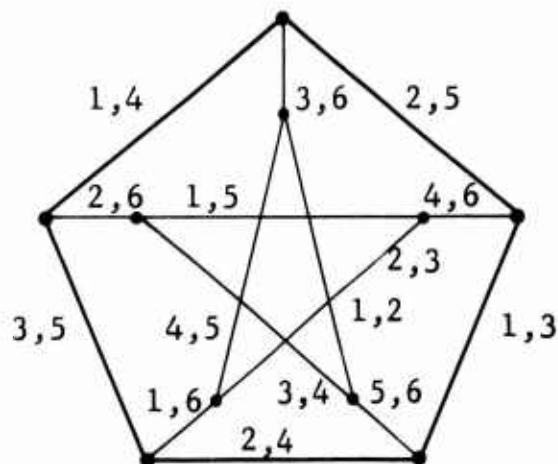have a "Tait bicoloring" in six colors?  The Petersen graph
does (see Fig. 12).



Fig. 12.

## 9.3. Sets of Fixed Cardinality Independent in Two Matroids

Let $M_1$ and $M_2$ be two matroids on the same set $E = \{1, \ldots, n\}$, having rank functions $r_1$ and $r_2$, respectively. Let A be the incidence matrix of all k-sets of E that are independent in both $M_1$ and $M_2$. Our last question concerns the blocking matrix B of A.

For $X \subseteq E$, define $r_{1,2}(X)$ to be the minimum of $r_1(W) + r_2(X-W)$ taken over all $W \subseteq X$. Do the inequalities in nonnegative variables $\xi_j$, $j \in E$,

$$(9.2) \qquad \sum_{j \in E-X} \xi_j \geq k - r_{1,2}(X), \qquad \text{all } X \subseteq E,$$

have just the rows of A as extreme solutions? It can be shown that the system (9.2) is a common generalization of the blocking matrix for permutations (Sec. 4.1) and of the blocking matrix for spanning arborescences having a common root (Sec. 4.4). Inequalities (9.2) are also closely connected with results of [20] on common partial transverasls of two families of sets.

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